Nonlinear Free Vibration of Single-Walled Carbon Nanotubes Embedded in Viscoelastic Medium Based on Asymptotic Perturbation Method

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ABSTRACT

In this paper, the nonlinear vibration of single-walled carbon nanotubes (SWCNTs) embedded in a Kelvin-Voigt foundation is studied. The SWCNT is considered as an elastic Euler-Bernoulli beam with von-Kármán type geometrical nonlinearity. Then, the governing equation of motion is derived based on the Hamilton’s principle. The nonlocal elastic field theory is utilized to introduce the small-scale effect into the equation of motion. Using the Galerkin method, the equation of motion is reduced to a nonlinear ordinary differential equation and solved by an asymptotic perturbation method called Krylov-Bogolubov-Mitropolskij (KBM) method. Two analytical formulas for the fundamental frequency and displacement field are derived. The obtained results for the case of simply-supported boundary conditions are reported and the effects of amplitude, residual stresses, and viscoelastic foundation are addressed and discussed.

Keywords: Single-walled carbon nanotubes, Nonlinear vibration, Nonlocal elastic field theory, Asymptotic perturbation method

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1 INTRODUCTION

In the past decade, it has been proved that the discovery of carbon nanotubes (CNTs) [1] due to their extraordinary electronic and mechanical properties brings about a great revolutionary in nanotechnology and numerous theoretical and experimental studies have been performed to predict the characteristics and behavior of these nanostructures.

In several studies, the potential application of CNTs as; mass detectors [2], nanosensors [3], nanoresonators [4-6], and nano-pipe conveyors of liquid or gas [7-10] have been investigated. Also, several researchers have examined the static and dynamic behavior of CNTs by using the analytical methods such as; molecular dynamics and molecular mechanics. However, the molecular dynamics simulations are not computationally efficient, especially for the large-scale molecular systems [11]. Also, experimental tests of nano-scale materials are too complicated to perform. As a result, recently, continuum-based models are widely considered to characterize the mechanical behavior of CNTs [12-14]. But, the classical continuum-based models are not able to take into account the small-scale effects [15]. Eringen [16, 17] proposed a nonlocal continuum theory which includes the small-scale features of material and has been widely implemented to model the nanostructures [18-22]. The residual stress which is produced during the fabrication processing is another important parameter that must be considered in design and analysis of micro- and nano-scale devices [23, 24]. The CNTs can be used as biological sensors [25] and to this aim it is essential that they become embedded with a biological soft tissue. The Kelvin-Voigt model is used to simulate the viscoelastic properties of biological soft tissues [26].

A comprehensive study on the nonlinear dynamic behavior of micro-beams using the classical continuum-based models can be found in the literature [27-32]. In contrast, a little investigation on the nonlinear dynamic behavior of nanostructures has been addressed in the past research works. Fu et al. [33] studied the nonlinear free vibration of single- and multi-walled CNTs embedded in elastic medium using a classical nonlinear elastic model. They utilized the incremental harmonic balanced method to solve the nonlinear equations of motion. Their results show that the surrounding elastic medium has significant impact on the fundamental natural frequency of multi-walled CNTs. This research work also stated that the fundamental natural frequency of CNTs increases while the elastic medium stiffness increases. Aminikhah and Hemmatnezhad [34] studied the nonlinear vibration of multi-walled CNTs based on the classical Euler-Bernoulli beam model with various boundary conditions. They used the homotopy analysis method to solve the nonlinear governing equations of motion. The nonlinear free vibration of functionally graded (FG) nano-beams has been studied using a nonlocal Euler-Bernoulli beam model with von-Kármán type geometrical nonlinearity [35]. Wang and Li [36] proposed a nonlinear nonlocal elastic model to study the nonlinear free vibration of SWCNTs embedded in a viscoelastic foundation with simply-supported boundary conditions. Then, they used the multiple scale method (MSM) to derive a zero-order analytical solution of nonlinear ordinary differential equation of motion. Their results show that the Winkler-type elastic stiffness of foundation has significant impact on the nonlinear vibration.

In this study, an analytical model is proposed based on the nonlocal elastic field theory to characterize the nonlinear vibration of embedded SWCNTs. The biological soft tissue is considered as a Kelvin-Voigt foundation (see Fig. 1). The KBM method is utilized to solve the reduced nonlinear equation of motion. Considering the small-scale effect, residual stresses, and various boundary conditions make the present model as a suitable choice to investigate the nonlinear dynamic behavior of embedded SWCNTs.
2 NONLOCAL ELASTIC FIELD THEORY

As mentioned above, the classical continuum-based models are unable to capture the small-scale effects [15]. In the nonlocal elastic field theory proposed by Eringen [16], unlike the classical continuum theory, the small-scale effects are taken into account by defining that the stress at each point is a function of the strain at all points in the domain. Based on the Eringen’s theory, the constitutive relation of a one-dimensional elastic isotropic material is stated as

$$\left(1 - (e_0^a)^2 \frac{d^2}{dx^2}\right)\sigma_{xx} = \sigma_{xx}^C$$

(1)

Where $\sigma_{xx}^C$, $\sigma_{xx}$ are the classical and nonlocal normal stress, respectively. Also, $a$ is an internal characteristic length and $e_0$ is a material constant called the scaling parameter which can be determined from matching the experimental and lattice dynamics simulation data. Multiplying Eq. (1) once by $z$ and then integrating over the cross-section area yields the following relation

$$M - (e_0^a)^2 M_{xx} = M^C$$

(2)

Where $M^C$ and $M$ are the classical and nonlocal sectional bending moments, respectively.

3 NONLINEAR VIBRATION MODEL

The kinetic energy, $T$, and elastic strain energy, $U$, as well as the virtual non-conservative work done by dissipative damping force, $\delta W_{nc}$, can be stated as
\[ T = \frac{1}{2} \int_{0}^{L} \rho A \left( w_{x}^{2} + u_{x}^{2} \right) dx \]  
(3a)

\[ U = \frac{1}{2} \int_{0}^{L} \left( M w_{xx} + N_{x} \varepsilon_{0} \right) dx + \frac{1}{2} \int_{0}^{L} \left( k_{w} w^{2} \right) dx \]  
(3b)

\[ \overline{\delta W_{nc}} = - \int_{0}^{L} c w_{,t} \delta w dx \]  
(3c)

Where \( u \) and \( w \) are the axial and transverse displacements, respectively. \( k_{w} \) and \( c \) are the Winkler-type stiffness and viscous damping coefficient of the Kelvin-Voigt foundation. \( L \) and \( \rho A \) are in order the axial length and mass per length of SWCNTs. \( N_{x} \) is the axial tension. \( \varepsilon_{0} \) is the nonlinear extensional strain which is related to the elastic displacement field and by considering the von-Kármán type geometrical nonlinearity can be expressed as

\[ \varepsilon_{0} = u_{,x} + \frac{w_{,xx}^{2}}{2} \]  
(4)

Using the Hamilton’s principle we have

\[ \delta H = \int_{t_{1}}^{t_{2}} \left( \delta T - \delta U + \overline{\delta W_{nc}} \right) dt = 0, \quad t_{1} < t < t_{2} \]  
(5)

After taking the required integration by parts, the equations of motion of an embedded SWCNT are obtained as:

\[ - \rho Aw_{,tt} - M_{,tx} + \left( N_{x} w_{,x} \right)_{,x} - k_{w} w - c w_{,t} = 0 \]  
(6a)

\[ \rho Au_{,tt} - \left( N_{x} \right)_{,x} = 0, \]  
(6b)

For the slender SWCNTs and small rotations, based on the Kirchhoff’s hypothesis [37], one can neglect the axial inertia. Thus, considering Eq. (6b), the axial tension \( N_{x} \) is constant and can be obtained as

\[ \int_{0}^{L} N_{x} dx = \int_{0}^{L} EA \varepsilon dx = \int_{0}^{L} EA \left( u_{,x} + \frac{w_{,xx}^{2}}{2} \right) dx \quad \Rightarrow \quad N_{x} = N_{0} + \frac{EA}{2L} \int_{0}^{L} w_{,xx}^{2} dx \]  
(7a)

Where

\[ N_{0} = \frac{EA}{L} \int_{0}^{L} u_{,x} dx \]  
(7b)

The \( N_{0} \) can be considered as residual force. Eq. (6a) is rewritten in the following form
By derivation Eq. (2) twice with respect to \( x \), and considering the classical bending moment as 
\[ M^c = EI w_{,xx}, \]
one can obtain 
\[ M_{,xx} - (e_0a)^2 (M_{,xx})_{,xx} = EI w_{,xxxx} \tag{9} \]

Rearranging Eq. (8) and then substituting it into Eq. (9) yields the following nonlocal equation of motion

\[
EI w_{,xxxx} + \rho Aw_{,tt} - \left( N_0 + \frac{EA}{2L} \int_0^L w_{,x}^2dx \right) w_{,xx} + k_w w + cw_{,t} \nonumber
\]

\[
- (e_0a)^2 \left[ \rho Aw_{,tt} - \left( N_0 + \frac{EA}{2L} \int_0^L w_{,x}^2dx \right) w_{,xxxx} + k_w w_{,xx} + c w_{,t} \right] = 0
\tag{10} \]

For the sake of simplicity and generality, the following dimensionless parameters are introduced as

\[
w = \frac{w}{L\gamma^2}, \quad \xi = \frac{x}{L}, \quad \gamma = \frac{r}{L}, \quad r^2 = \frac{I}{A}, \quad \tau = \frac{(E/\rho)^{1/2}}{L} t,
\]

\[
\bar{k}_w = \frac{k_w}{EA\gamma^2}, \quad \bar{N}_0 = \frac{N_0}{EA\gamma^2}, \quad \mu = \frac{e_0a}{L},
\]

\[
\zeta = \frac{c}{2\rho A\omega_0}, \quad \bar{\alpha} = \frac{\omega_0}{\left( EI/\rho AL^4 \right)^{1/2}}.
\tag{11} \]

Where \( \omega_0 \) is the classical first mode natural frequency. By inserting the dimensionless parameters into Eq. (10), the dimensionless form of it can be represented as

\[
w_{,xxxx} + w_{,tt} - \left( \bar{N}_0 + \frac{\gamma^2}{2} \int_0^1 \bar{w}_{,x}^2dx \right) w_{,xx} + \bar{k}_w w + 2\zeta \bar{\alpha} w_{,t} \nonumber
\]

\[
- \mu^2 \left[ w_{,xxxx} - \left( \bar{N}_0 + \frac{\gamma^2}{2} \int_0^1 \bar{w}_{,x}^2dx \right) \bar{w}_{,xxxx} + \bar{k}_w \bar{w}_{,xx} + 2\zeta \bar{\alpha} \bar{w}_{,t} \right] = 0
\tag{12} \]

Using the single-mode approximation, the solution of Eq. (12) can be considered as

\[
w(\xi, \tau) = W(\tau) \phi(\xi)
\tag{13} \]

Where \( \phi(\xi) \) is the nonlocal first natural mode shape. The general expression for the nonlocal natural mode shapes is represented as [38]
\[ \phi(\xi) = C_1 \sin(\sqrt{\alpha} \kappa_1 \xi) + C_2 \cos(\sqrt{\alpha} \kappa_1 \xi) + C_3 \sinh(\sqrt{\alpha} \kappa_2 \xi) + C_4 \cosh(\sqrt{\alpha} \kappa_2 \xi) \]  

where the constants \( C_1 - C_4 \) are obtained by inserting the boundary conditions. \( \kappa_1 \) and \( \kappa_2 \) have been defined as

\[
\left\{ \kappa_1 \right\} = \sqrt{\frac{\mu^2 + 4 \pm \sqrt{\mu^4 - 4\mu^2 + 16\mu^2}}{2}} \quad (15)
\]

By substituting Eq. (13) into Eq. (12) and using the Galerkin method, Eq. (12) reduces to

\[
\left( \Gamma_{14} - (\Gamma_{12} - \mu^2 \Gamma_{14}) \kappa_0 + (\Gamma_{10} - \mu^2 \Gamma_{10}) \kappa_w \right) W + (\Gamma_{10} - \mu^2 \Gamma_{12}) W_{rr} 
+ (\Gamma_{10} - \mu^2 \Gamma_{12}) 2 \zeta_0 \alpha W_{r} - (\Gamma_{12} - \mu^2 \Gamma_{14}) \gamma^2 \overline{N}_{nl} W^3 = 0 
\]  

(16)

Where

\[
\Gamma_{14} = \int_0^1 \phi \phi \delta_{\xi} \xi \, d\xi, \quad \Gamma_{12} = \int_0^1 \phi \phi \delta_{\xi} \, d\xi, \quad \Gamma_{10} = \int_0^1 \phi \phi \xi \, d\xi, \quad \overline{N}_{nl} = \frac{1}{2} \int_0^1 \phi \phi \xi \, d\xi 
\]  

(17a-d)

By rearranging and manipulating Eq. (16), it can be compacted as

\[ W_{rr} + 2 \zeta_0 \alpha W_{r} + \Omega_0 W + \varepsilon \overline{C}_{nl} W^3 = 0 \]  

(18)

Where

\[
\Omega_0 = \frac{\Gamma_{14} - (\Gamma_{12} - \mu^2 \Gamma_{14}) \kappa_0 + \kappa_w}{(\Gamma_{10} - \mu^2 \Gamma_{10})} - \frac{2 \zeta_0 \alpha}{(\Gamma_{12} - \mu^2 \Gamma_{14})} \overline{N}_{nl}, \quad \varepsilon = \gamma^2 
\]  

(19a-c)

For the slender SWCNTs \( \varepsilon \) is a small term, i.e. \( \varepsilon \ll 1 \). Also, for a biological soft tissue with weakly viscoelastic damping behavior one can considered the damping coefficient as \( \zeta = \varepsilon \zeta \). Thus, Eq. (18) can be expressed as

\[ W_{rr} + \Omega_0 W = -\varepsilon \left( 2 \zeta_0 \alpha W_{r} + \overline{C}_{nl} W^3 \right) \]  

(20)

The asymptotic perturbation KBM method [39] is used to solve Eq. (20). Based on the KBM method, the amplitude and phase of vibration are time-varying terms and the solution of Eq. (20) is approximated as following perturbation series

\[ W(\tau) = a \cos \psi + \sum_{k=1}^{K} \varepsilon^{k} W_k a(\tau) \psi(\tau) + O(\varepsilon^{K+1}) \]  

(21)
Where \( a \) and \( \psi \) are the amplitude and phase of vibration, respectively. Also, the additional series for the derivative of amplitude and phase with respect to time has been introduced as

\[
\dot{a}_x = \sum_{k=1}^{K} \varepsilon^k A_k \left[ \dot{a}(\tau) \right] + O(\varepsilon^{K+1}) \quad (22a)
\]

\[
\dot{\psi}_x = \Omega_0 + \sum_{k=1}^{K} \varepsilon^k \Omega_k \left[ \dot{a}(\tau) \right] + O(\varepsilon^{K+1}) \quad (22b)
\]

From Eq. (21)

\[
W_{x} = \ddot{a}_x \cos \psi - \dot{a}_x \dot{\psi}_x \sin \psi + \sum_{k=1}^{K} \varepsilon^k \left( W_{k,\ddot{a}} \ddot{a}_x + W_{k,\dot{\psi}} \dot{\psi}_x \right) + O(\varepsilon^{K+1}) \quad (23a)
\]

\[
W_{\tau x} = \ddot{a}_x \cos \psi - 2 \dot{a}_x \dot{\psi}_x \sin \psi - \ddot{\psi}_x \sin \psi - \dot{\psi}_x^2 \cos \psi
\]

\[
+ \sum_{k=1}^{K} \varepsilon^k \left( \dot{W}_{k,\ddot{a}} a_x + 2 \dot{W}_{k,\dot{\psi}} a_x \psi_x x + W_{k,\ddot{a}} \ddot{a}_x + W_{k,\dot{\psi}} \dot{\psi}_x x + W_{k,\dot{\psi}} \psi_x x^2 + W_{k,\ddot{\psi}} \ddot{\psi}_x x \right) + O(\varepsilon^{K+1}) \quad (23b)
\]

According to Eqs. (22a) and (22b), the second-order approximation of term \( \dot{\psi}_{x\tau} \) can be expressed as

\[
\dot{\psi}_{x\tau} = \sum_{k=1}^{K} \varepsilon^k \Omega_{k,\ddot{a}} \ddot{a}_x = \sum_{k=1}^{K} \varepsilon^k \Omega_{k,\dot{\psi}} \dot{\psi}_x x + \sum_{k=1}^{K} \varepsilon^k A_k = \varepsilon^2 \Omega_{k,\ddot{a}} A_d + O(\varepsilon^3) \quad (24)
\]

By substituting the relations (23) and (24) into Eq. (20) and then rearranging the terms with same order with together and eliminating the higher order terms, the following recurrent set of differential equations can be derived

\[
\varepsilon^1:
\]

\[
W_{1,\dot{\psi}\psi} + W_1 = \frac{1}{\Omega_0^2} \left( F_0 + 2A_1 \Omega_0 \sin \psi + 2\Omega_1 \Omega_0 \ddot{a} \cos \psi \right) \quad (25a)
\]

\[
\varepsilon^2:
\]

\[
W_{2,\dot{\psi}\psi} + W_2 = \frac{1}{\Omega_0^2} \left( F_1 + 2A_2 \Omega_0 \sin \psi + 2\Omega_2 \Omega_0 \ddot{a} \cos \psi \right) \quad (25b)
\]

Where

\[
F_0 = 2 \ddot{a} \Omega_0 \ddot{a} \sin \psi - \ddot{a} a \cos^3 \psi \quad (26a)
\]
\[ F_i = -2\Omega_1 \Omega_0 W_{1,\psi} - 2A_i A_0 W_{1,\theta} - \left( A_i A_{1,\theta} - \Omega_0^2 \bar{a} \right) \cos \psi + \left( 2A_i \Omega_1 + \bar{a} A_i \Omega_{1,\theta} \right) \sin \psi \]
\[ - 3C_{nL} \bar{a}^2 W_i \cos^2 \psi - 2\bar{\zeta} \alpha \left( A_i \cos \psi - \bar{a} \Omega_1 \sin \psi + \Omega_0 W_{1,\psi} \right) \]  

(26b)

Solving the first-order equation yields

\[ W_i = \frac{\bar{C}_{nL} \bar{a}^2}{32\Omega_0^2} \cos \psi, \quad A_i = -\bar{\alpha} \bar{\zeta} \bar{a}, \quad \Omega_1 = \frac{3\bar{C}_{nL} \bar{a}^2}{8\Omega_0} \]  

(27)

According to series (22a) and (22b) for the first-order approximation we have

\[ \bar{a}_x = \varepsilon A_i = -\varepsilon \bar{\alpha} \bar{\zeta} \bar{a} \]

(28a)

\[ \psi_x = \Omega_0 + \varepsilon \Omega_1 = \Omega_0 + \varepsilon \frac{3\bar{C}_{nL} \bar{a}^2}{8\Omega_0} \]  

(28b)

Firstly, Eq. (28a) is solved as

\[ a = a_0 e^{-\bar{\zeta} \tau} \]  

(29)

Then by inserting Eq. (29) into Eq. (28b) and solving it, the following relation for the phase can be achieved as

\[ \psi = \Omega_0 \tau - \varepsilon \frac{3\bar{C}_{nL} a_0}{8\Omega_0} e^{-2\bar{\zeta} \tau} + \phi_0 \]  

(30)

Finally, the displacement field of SWCNTs embedded in a biological soft tissue considering the first-order approximation is derived as

\[ w(\bar{\xi}, \tau) = \left[ \frac{\bar{a}_0 e^{-\bar{\lambda} \tau} \cos \left( \Omega_0 \tau - \varepsilon \frac{B}{2\bar{\lambda}} e^{-2\bar{\lambda} \tau} + \phi_0 \right)}{\bar{a}_0 e^{-\bar{\lambda} \tau} \cos \left( \Omega_0 \tau - \varepsilon \frac{B}{2\bar{\lambda}} e^{-2\bar{\lambda} \tau} + \phi_0 \right)} \right] \phi(\bar{\xi}) \]  

(31)

Where

\[ \bar{B} = \frac{3\bar{C}_{nL} a_0^2}{8\Omega_0}, \quad \bar{Y} = \frac{\bar{C}_{nL} a_0}{32\Omega_0^2}, \quad \bar{\lambda} = \bar{\alpha} \bar{\zeta} \]  

(32)

The \( \bar{a}_0 \) and \( \phi_0 \) can be obtained from initial conditions.
4 RESULTS

To investigate the effect of amplitude, nonlocal parameter, residual forces and viscoelastic medium on the nonlinear vibration of embedded SWCNTs, a zigzag (15, 0) SWCNT with simply-supported boundary condition at both ends has been considered. The mechanical properties of the SWCNT is introduced as:

- density $\rho = 2300 \, \text{kg/m}^3$
- Young’s modulus $E = 1.0 \, \text{TPa}$
- diameter $D = 1.18 \, \text{nm}$
- wall thickness $t = 0.34 \, \text{nm}$
- axial length $L = 10 \, \text{nm}$

As mentioned above, the nonlocal elastic field theory has been widely used to survey the mechanical behaviors of nanostructures and the range of the nonlocal parameter $e_0a$ has been considered as $0.0 \leq e_0a \leq 2.0 \, \text{nm}$ [18]. The internal characteristic length is represented as $a = 0.1421 \, \text{nm}$ which is the length of a carbon-carbon covalent bond in the atomistic lattice of SWCNTs.

The mechanical properties of the Kelvin-Voigt foundation for a biological soft tissue are considered as $\omega_0 = 0.1 \, \text{MPa}$, $c = 1.02 \times 10^{-4} \, \text{Pa.s}$ [26]. Considering the simply-supported boundary conditions, the single-mode approximation of the displacement field is

$$w(\xi, \tau) = W(\tau) \sin(\pi \xi)$$

Consequently, using Eq. (19a, b), we have

$$\Omega_0^2 = \frac{\pi^4}{1 + \pi^2 \mu} + \pi^2 \tilde{N}_0 + \tilde{k}_w, \quad \tilde{C}_{nl} = \frac{\pi^4}{4}$$

In the case of simply-supported boundary conditions, it is to be found that the nonlinear term $\tilde{C}_{nl}$ is not affected by the nonlocal parameter. In other word, the nonlocal parameter just affect the natural frequency term $\Omega_0^2$. Also, the dimensionless fundamental frequency of a simply-supported beam is $\bar{\alpha} = \pi^2$. Based on Eq. (18) and Eq. (34), the reduced nonlinear equation of motion is

$$W_{\tau\tau} + 2\zeta \pi^2 W_{\tau} + \Omega_0^2 W + \varepsilon \frac{\pi^4}{4} W^3 = 0$$

As mentioned above, it is considered that the SWCNT lies in a biological soft tissue with the small viscous damping coefficient, i.e. $\zeta = \varepsilon \pi^2$. Therefore, based on Eq. (31), the displacement field of the embedded SWCNT can be formulated as

$$w(\xi, \tau) = \left[ -a_0 e^{-\pi^2 \xi^2} \cos \left( \Omega_0 \tau - \frac{3\pi^2 a_0}{64 \zeta \Omega_0} e^{-2\pi^2 \xi^2} \xi^2 + \varphi_0 \right) ight. \\ + \varepsilon \frac{\pi^4}{128 \Omega_0^2} e^{-3\pi^2 \xi^2} \cos 3 \left( \Omega_0 \tau - \frac{3\pi^2 a_0}{64 \zeta \Omega_0} e^{-2\pi^2 \xi^2} \xi^2 + \varphi_0 \right) \left. \right] \sin(\pi \xi)$$

By considering the initial condition for the phase as $w_{\tau=0} = 0$, Eq. (36) can be rewritten as
\[
\begin{align*}
    w(x, t) = &\left[ -\tilde{a}_0 e^{-\pi^2\xi^2} \cos \left( \Omega_0 t - \epsilon \frac{3\pi^2\tilde{a}_0^2}{64\xi^2\Omega_0} \left( e^{-2\pi^2\xi^2} - 1 \right) \right) \\
    &+ \frac{\pi^4 \tilde{a}_0^2}{128\Omega_0^3} e^{-3\pi^2\xi^2} \cos 3 \left( \Omega_0 t - \epsilon \frac{3\pi^2\tilde{a}_0^2}{64\xi^2\Omega_0} \left( e^{-2\pi^2\xi^2} - 1 \right) \right) \sin \left( \pi \xi \right) \right]
\end{align*}
\]

The \( \tilde{a}_0 \) can be obtained by the initial displacement.

### 4.1 Validation

In order to validate the solution, the problem is solved for two cases of the mid-span initial displacements \( w_0 = 1 \text{nm} \) and \( w_0 = 0.5 \text{nm} \). Then, for each case the influence of nonlocal parameter is addressed. Using the computer package MATLAB, for each case, the nonlinear differential equation (35) is solved by means of Runge-Kutta method. Also, the obtained results from the present model and Runge-Kutta method are compared with those of obtained by Wang and Li’s analytical relation [36] which has been derived based on the MSM. The time history of the mid-span displacement for the initial displacement \( w_0 = 1 \text{nm} \) and nonlocal parameters \( e_0a = 0.0 \text{nm} \) and \( e_0a = 2.0 \text{nm} \) is illustrated in Figs. 2a and 2b, respectively. Also, the corresponding phase plots for these cases are depicted in Figs. 3a and 3b, respectively. From Figs. 2a and 2b it can be found that there is a little difference between the present model and Runge-Kutta numerical solution for the case of initial displacement \( w_0 = 1 \text{nm} \), but the larger deviation between the MSM [36] and both of the KBM method and Runge-Kutta is observed. The corresponding phase plots of trajectories give the more obvious view about the accuracy of these approaches to predict the nonlinear dynamic behavior of the embedded SWCNTs. The results show that the nonlocal parameter has a negative impact on the accuracy of the analytical solutions. In addition, this scenario is repeated for the initial displacement \( w_0 = 0.5 \text{nm} \). As it can be seen from Figs. 4a and 4b and their corresponding phase plots (Fig. 5), both of the KBM method and MSM are in good agreement with the Runge-Kutta numerical solution. In this case, the nonlocal parameter has a little impact on the accuracy of the obtained results relative to each other. Also, the comparison of the results for two cases of \( w_0 = 1 \text{nm} \) and \( w_0 = 0.5 \text{nm} \) shows that the precision of the approximate analytical formula depends on the amplitude of vibration. By decreasing the amplitude of the vibration, the accuracy of the results increases. The nonlocal nonlinear elastic beam model, especially in the case of large amplitude vibrations, requires higher order solution in the asymptotic perturbation methods compared with the classical nonlinear elastic beam model.
Fig. 2. The time history of displacements for the case of $w_0 = 1\text{nm}$, $k_w = 0.1\text{MPa}$, $c = 1.02 \times 10^{-4}\text{Pa.s}$; 
(a) $e_0a = 0.0\text{nm}$ and $e_0a = 2.0\text{nm}$.

Fig. 3. The phase plot of trajectories for the case of $w_0 = 1\text{nm}$, $k_w = 0.1\text{MPa}$, $c = 1.02 \times 10^{-4}\text{Pa.s}$; (a) $e_0a = 0.0\text{nm}$ and (b) $e_0a = 2.0\text{nm}$.
Fig. 4. The time history of displacements for the case of \( w_0 = 0.5 \text{nm} \), \( k_w = 0.1 \text{MPa} \), \( c = 1.02 \times 10^{-4} \text{ Pa.s} \):

(a) \( e_0 a = 0.0 \text{nm} \) and (b) \( e_0 a = 2.0 \text{nm} \).

Fig. 5. The phase plot of trajectories for the case of \( w_0 = 0.5 \text{nm} \), \( k_w = 0.1 \text{MPa} \), \( c = 1.02 \times 10^{-4} \text{ Pa.s} \):

(a) \( e_0 a = 0.0 \text{nm} \) and (b) \( e_0 a = 2.0 \text{nm} \).
4.2 Nonlinear dynamic analysis

The nonlinear dynamic behavior of an embedded SWCNT is affected by various parameters. Here, the effects of viscoelastic foundation, nonlocal parameter and residual force have been investigated. As mentioned before, the SWCNTs are embedded in a biological soft tissue. In order to examine the influence of viscous damping and elastic stiffness of surrounding medium, the time history of displacement response and the phase plot for the cases of $k_w = 0.0 \text{MPa}$, $k_w = 0.1 \text{MPa}$ and $k_w = 1.0 \text{GPa}$ are plotted in Figs. 6a and 6b, respectively. The results are obtained for the mid-span initial displacement $w_0 = 0.5 \text{nm}$, damping coefficient $c = 1.02 \times 10^{-4} \text{Pa.s}$ and nonlocal parameter $e_o a = 2.0 \text{nm}$. As it can be found from the results, the viscoelastic foundation dissipates the mechanical energy of the system and after a certain time from the initial point of the vibration, the amplitude tends to zero. In this case, the amplitude from the initial value $w_0 = 0.5 \text{nm}$ reaches to $w = 86 \times 10^{-3} \text{nm}$ after $t = 1 \times 10^{-10} \text{s}$ from the initial point of the vibration. Also, the results show that the elastic stiffness of a biological soft tissue has a little impact on the nonlinear dynamic behavior of embedded SWCNTs and it can be neglected. The nonlinear vibration of SWCNTs embedded in a stiffer viscoelastic foundation ($k_w = 1.0 \text{GPa}$) is surveyed and obtained results are compared with the case of biological soft tissue. This comparison reveals that the elastic stiffness of the surrounding medium especially for the large values of the stiffness increases the velocity of the vibration.

In order to study the influence of nonlocal parameter on the nonlinear vibration of SWCNTs embedded in viscoelastic medium, the time history of displacement response and corresponding phase plot for three different nonlocal parameters $e_o a = 0.0 \text{nm}$, $e_o a = 1.0 \text{nm}$ and $e_o a = 2.0 \text{nm}$ are illustrated in Figs. 7a and 7b, respectively. It should be noted that by considering the nonlocal parameter as $e_o a = 0.0 \text{nm}$, the nonlocal nonlinear elastic beam model converts to the classical nonlinear elastic beam model. The results show that the nonlocal parameter could change the nonlinear dynamic behavior of embedded SWCNTs and disregarding it might lead to a remarkable error in our analysis. From Figs. 7a and 7b, the velocity of the nonlinear vibration decreases while that the nonlocal parameter increases and this effect becomes more intense for the larger values of the nonlocal parameter. Therefore, the classical nonlinear elastic beam model overestimates the velocity of the vibration in the case of short length SWCNTs.

As another important parameter, the effect of residual force on the nonlinear vibration is studied. The residual force might be exerted on the embedded SWCNTs which have utilized as biosensors due to fabrication processing [24]. In order to design a more reliable nanostructure system, it is essential to measure the residual forces and take them into account in further analysis. The results for two cases of the residual tensile force $N_0 = 10 \text{nN}$ and residual compressive force $N_0 = -10 \text{nN}$ are obtained and compared with those of $N_0 = 0.0 \text{nN}$ (see Figs. 8a and 8b). It is observed that the residual tensile force increases the velocity of the nonlinear vibration and inversely the residual compressive force reduces it. As a result, the residual force in the CNT-based sensors and devices must be obtained and considered in the nonlinear dynamic behavior analysis. Also, the residual force can be implemented as a controller of nonlinear dynamic behavior for the fixed dimension embedded SWCNTs.
Fig. 6. The effect of elastic stiffness of surrounding medium for the case of $w_0 = 0.5\, nm$, $c = 1.02 \times 10^{-4}\, Pa.s$; (a) time history of displacements and (b) phase plot of trajectories.

Fig. 7. The effect of nonlocal parameter for the case of $w_0 = 0.5\, nm$, $k_w = 0.1\, MPa$, $c = 1.02 \times 10^{-4}\, Pa.s$; (a) time history of displacements and (b) phase plot of trajectories.
Fig. 8. The effect of residual force for the case of $w_0 = 0.5 \text{nm}$, $k_w = 0.1 \text{MPa}$, $c = 1.02 \times 10^{-4} \text{ Pa.s}$; (a) time history of displacements and (b) phase plot of trajectories.

5 CONCLUSIONS

A continuum-based model for the nonlinear vibration of SWCNTs embedded in a Kelvin-Voigt foundation is presented based on the nonlocal elastic field theory. Then, using the KBM method the nonlocal nonlinear equation of motion has been solved. For case study, the nonlinear vibration of an embedded SWCNT with simply-supported boundary conditions is investigated and the effects of amplitude, viscoelastic foundation, nonlocal parameter and residual force are studied. The presence of Kelvin-Voigt foundation depends on its elastic stiffness and viscous damping parts could has various effects on the nonlinear dynamic behavior of embedded SWCNTs. The residual tensile/compressive force intends to raise/reduce the velocity of the nonlinear vibration. Moreover, the nonlocal parameter decreases the velocity of the vibration. These results can be useful in the design process of biological nanosensors and nanoscale devices embedded in the viscoelastic medium.

REFERENCES


